Problem Set 1: Quasi-circular Inspirals **PHYS-F484**

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1 Quadrupolar Gravitational Radiation

This problem set will focus on the lowest-order Newtonian approximation to quasi-circular inspirals (QCIs). Let us briefly review what we will need for this exercise session and lay the foundations that will then be applied to QCIs. We work with a perturbed flat metric, that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $h_{\mu\nu}$ is considered small with respect to the flat Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{2}$$

This allows one to work in a linearized regime where orders of h_{ij}^2 and beyond are neglected, in which the Einstein equations explicitly reduce to a wave equation for the metric perturbation. A convenient gauge choice, the *transverse-traceless* gauge (TT-gauge), shows that this metric perturbation only contains two physical, propagating, degrees of freedom, as expected from the representation theory of massless particles. The metric perturbation, in this gauge, can be expressed as

$$h_{ij}^{TT}(t, \mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \dots\right]_{ret.}, \tag{3}$$

where $\Lambda_{ij,kl}$ is the TT-gauge projector

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}, \quad P_{ij} = \delta_{ij} - n_i n_j, \tag{4}$$

and

$$S^{ij,i_1\dots i_\ell}(t) = \int d^3x\, T^{ij}(t, \mathbf{x})x^{i_1}\dots x^{i_\ell}, \tag{5}$$

$T_{\mu\nu}$ being the matter stress-energy tensor. The subscript $_{ret.}$ indicates that all quantities must be evaluated at the retarded time $t - r/c$. The components T_{ij} of the stress-energy tensor depend on the fluxes of energy and momentum within the source itself and are therefore difficult to measure. Our strategy is to relate the moments of S^{ij} to moments of quantities that are easier to work with, such as the energy and linear momentum densities.

Problem 1.1. ***Conservation laws for stress-energy moments***
a) In linearized gravity, the conservation of the stress-energy tensor is expressed as $\partial_\mu T^{\mu\nu} = 0$. We define the energy density moments as

$$M^{i_1\dots i_\ell} = \frac{1}{c^2}\int d^3x\, T^{00}(t, \mathbf{x})x^{i_1}\dots x^{i_\ell} \tag{6}$$

and the linear momentum moments as

$$P^{j,i_1\dots i_\ell} = \frac{1}{c}\int d^3x\, T^{0j}(t,\mathbf{x})x^{i_1}\dots x^{i_\ell}. \tag{7}$$

Show that in a volume V enclosing the source entirely, we have

$$\dot{M}^{i_1\dots i_\ell} = \ell P^{(i_1,i_2\dots i_\ell)} \tag{8}$$

and

$$\dot{P}^{j,i_1\dots i_\ell} = \ell S^{j(i_1,i_2\dots i_\ell)}, \tag{9}$$

where we used the notation

$$P^{(i_1\dots i_l)} = \frac{1}{l!}\left((l-1)!P^{i_1\dots i_l} + (l-1)!P^{i_2 i_1\dots i_l} + \dots\right). \tag{10}$$

b) Use your results to express h_{ij}^{TT} in terms of the moments of $T^{0\mu}$ up to subleading order in $1/c$.

At leading order, we have thus shown that

$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t-r/c), \tag{11}$$

i.e. the leading order of gravitational radiation is quadrupolar. This is to be expected, as you have just shown that M and P^i are conserved in linearized gravity. Although they are not conserved when non-linearities are considered, the absence of monopolar and dipolar radiation holds in more general settings. We now want to unfold this expression for more practical purposes.

Problem 1.2. *Angular distribution of quadrupolar radiation*
a) Consider that the radiation is aligned with an axis of the detector frame $\{x,y,z\}$, let us say $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Show that the two physical polarisations are given by

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right), \quad h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}, \tag{12}$$

evaluated at $t - r/c$.

b) For $\hat{\mathbf{n}}$ not aligned with $\hat{\mathbf{z}}$, consider a new frame defined by three vectors $\{\mathbf{u},\mathbf{v},\mathbf{n}\}$ such that $\mathbf{u}\times\mathbf{v} = \mathbf{n}$ and $\mathbf{u}$ lies in the (x,y)-plane. With respect to the original frame, $\mathbf{n}$ forms an angle θ with the z axis and an angle ϕ with the y axis. How would you write $\mathbf{n}$ in the $\{x,y,z\}$ coordinates? What is the rotation matrix allowing you to go from one frame to the other?

c) Show that in the original frame, we have

$$\begin{aligned} h_+ = \frac{1}{r}\frac{G}{c^4}\Big[&\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) - \ddot{M}_{33}\sin^2\theta \\ &- \ddot{M}_{12}\sin 2\phi(1+\cos^2\theta) + \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big], \end{aligned} \tag{13}$$

$$h_\times = \frac{1}{r}\frac{G}{c^4}\left[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2(\ddot{M}_{12}\cos 2\phi\cos\theta - \ddot{M}_{13}\cos\phi\sin\theta + \ddot{M}_{23}\sin\phi\sin\theta)\right].$$

2 Quasi-circular inspirals

Before *plunging* into quasi-circular inspirals, let us consider a system of two point masses m_1 and m_2 whose relative coordinate performs a circular motion. We neglect the backreaction of gravitational wave emission on this system. Without any loss of generality, we can choose our axes such that the orbit lies in the (x, y)-plane. The relative coordinate $\mathbf{x_0} = \mathbf{x_1} - \mathbf{x_2}$ is given by

$$x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right), \quad y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right), \quad z_0(t) = 0, \tag{14}$$

with R the orbital radius.

Problem 2.1. *Show that the quadrupolar mass moment is given by*

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j. \tag{15}$$

In terms of the total mass $m = m_1 + m_2$, the reduced mass $\mu = \dfrac{m_1 m_2}{m}$ and the center of mass position $\mathbf{x}_{\mathrm{CM}}$, this last expression can be rewritten as

$$M^{ij} = m x_{\mathrm{CM}}^i x_{\mathrm{CM}}^j + \mu x_0^i x_0^j. \tag{16}$$

For an isolated system, the center of mass is not accelerated and one can choose the origin of our reference frame to coincide with it. Then, the mass multipole moment reduces to that of a particle of mass μ located at $\mathbf{x_0}$.

Problem 2.2. *a) Compute the mass moments. Show that*

$$h_+ = \frac{4G}{rc^4}\mu R^2\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{ret} + 2\phi), \; h_\times = \frac{4G}{rc^4}\mu R^2\omega_s^2\cos\theta\sin(2\omega_s t_{ret} + 2\phi). \tag{17}$$

We notice that the frequency of the radiation is twice the frequency of the circular motion.

b) Compute the angular distribution of the radiated power

$$\left(\frac{dP}{d\Omega}\right)_{quad} = \frac{r^2c^3}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle. \tag{18}$$

Notice that there is no angle for which no power is received. Integrate this distribution to find the total radiated power

$$P_{quad} = \frac{32G\mu^2R^4\omega_s^6}{5c^5}. \tag{19}$$

We now make this scenario a bit more precise. The total energy of the system is given by

$$E_{\mathrm{orbit}} = E_{\mathrm{kin.}} + E_{\mathrm{pot.}} = -\frac{Gm_1m_2}{2R}. \tag{20}$$

As energy is radiated, the orbital energy E_{orbit} decreases, and thus R must also decrease. Given Kepler's law

$$\omega_s^2 = \frac{Gm}{R^3}, \tag{21}$$

this leads to the increase of the inspiral frequency.

Problem 2.3. *Show that as long as $\dot{\omega}_s \ll \omega_s^2$, the radial velocity is negligible compared to the tangential velocity.*

This condition ensures that the system is in the *quasi-circular regime*. It is convenient to introduce the *chirp mass*

$$M_c = \mu^{3/5} m^{2/5}. \tag{22}$$

Problem 2.4. *Show that*

$$h_+ = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\pi f_{GW} t_{ret.} + 2\phi) \tag{23}$$

and

$$h_+ = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}}{c}\right)^{2/3}\cos\theta\sin(2\pi f_{GW} t_{ret.} + 2\phi). \tag{24}$$

Likewise, the radiated power can be rewritten in terms of the chirp mass as

$$P = \frac{32c^5}{5G}\left(\frac{GM_c\,\omega_{\rm GW}}{2c^3}\right)^{10/3}. \tag{25}$$

Problem 2.5. *Show that the gravitational radiation frequency obeys the equation*

$$\dot{f}_{GW} = C f_{GW}^{11/3}, \tag{26}$$

where C is a constant that depends on the parameters of the system. Solve this differential equation and discuss the regularity of $f_{GW}(t)$.

We call the integration constant obtained by solving this equation the *coalescence time* $t_{\rm coal.}$. In terms of $\tau = t_{\rm coal.} - t$, we have

$$f_{\rm GW}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}. \tag{27}$$

Introducing numerical values, this can be rewritten as

$$f_{\rm GW}(\tau) \simeq 134\,{\rm Hz}\left(\frac{1.21\,M_\odot}{M_c}\right)^{5/8}\left(\frac{1\,s}{\tau}\right)^{3/8}, \tag{28}$$

where we have taken $1.21\,M_\odot$ as the mass reference. This is the chirp mass of two $1.4\,M_\odot$ bodies.

Problem 2.6. *a) Using Eq. (27), discuss the evolution of the orbital radius with respect to time and the regime of validity of the quasi-circular approximation.*

b) To compute the waveform associated to the quasi-circular inspiral, all appearances of $\omega_{GW} t$ have to be replaced with

$$\Phi(t) = \int_{t_0}^{t} dt' \omega_{GW}(t'). \tag{29}$$

Compute $\Phi(t)$ and sketch the waveform associated to the inspiral.

Problem 2.7. ***The first gravitational wave event: GW150914***

In 2015, the first gravitational wave event, GW150914, was detected by LIGO. It was estimated that the two inspiralling bodies had masses 36.2 $M_\odot$ and 29.1 $M_\odot$.

a) Compute the gravitational wave frequency f_{GW} 0.02 seconds before the coalescence of the two bodies. What would the orbital radius of the system be at that moment? What can you then say about the nature of this two-body system?

b) Compute the energy carried away by the gravitational wave emission as the gravitational wave frequency increases from 10 to 30 Hz.

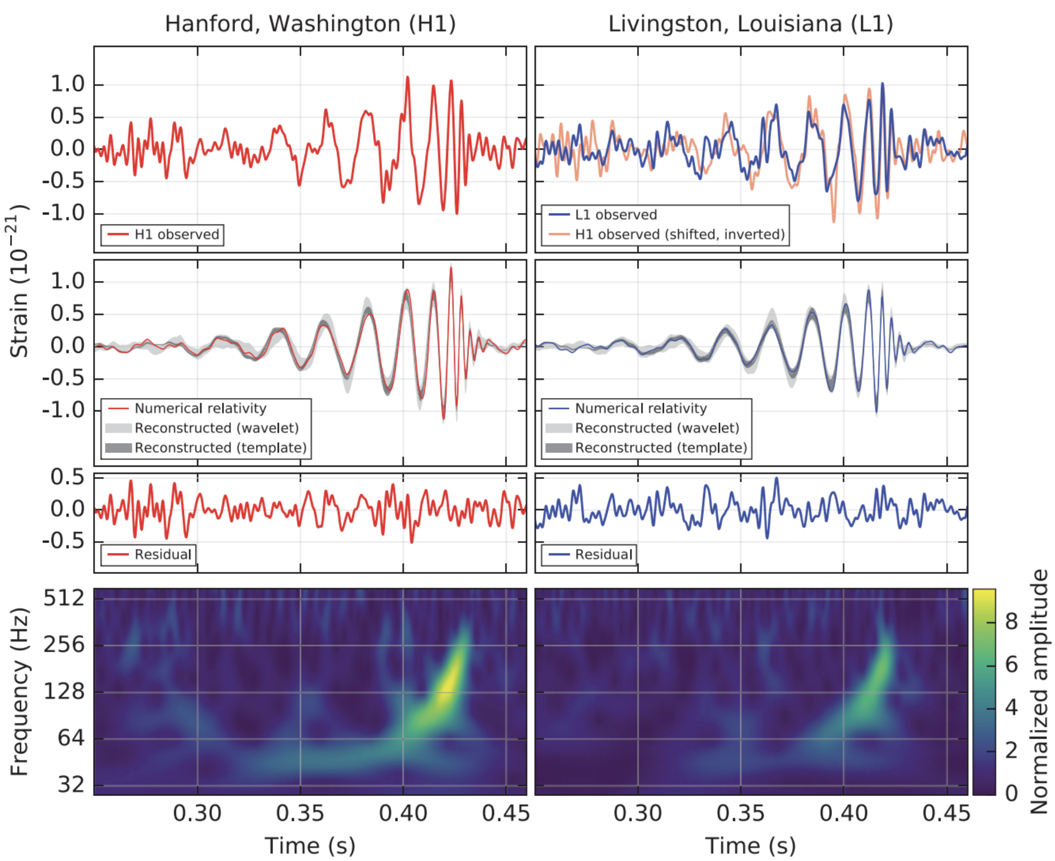

Figure 1: The gravitational wave event GW150914 observed by the LIGO Hanford (H1, left column panels) and Livingston (L1, right column panels) detectors. (arXiv:1602.03837 [gr-qc])